Supply function equilibrium with taxed benefits*

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Abstract

Supply function equilibrium models are used to study electricity market auctions. We study the effects on supply function equilibrium of a system tax on the observed benefits of suppliers. Such a tax provides an incentive for agents to submit more competitive offers. The model is extended to a setting in which the agents are taxed on the benefits accruing to them from a transmission line expansion (in order to help fund the line).

1 Introduction

The supply function equilibrium (SFE) is a natural concept to use in studying electricity pool markets. Here generators submit supply schedules in the form of increasing offer curves to an auction that dispatches generation to those suppliers offering at the lowest prices. The first use of SFE in this context was by Green and Newbery [6] in a study of the England and Wales electricity pool. The paper [6] was based on the theory of supply function equilibrium laid out by Klemperer and Meyer [11]. This theory has been extended to inelastic demand and pay-as-bid auctions by several authors, notably [1],[2],[3],[4],[5],[7],[12],[14]. As well as [6], a number of authors have applied SFE in a practical setting. For example [10] and [13] use the SFE model to study generator behaviour in the Texas electricity pool. The recent survey paper by Holmberg and Newbery [9] gives a good overview of the state of the art in SFE models.

In this paper we examine a supply-function equilibrium model in circumstances where suppliers are taxed on the benefits they earn. At first sight, paying a (non-progressive) tax that is a fixed proportion of profits will not alter agent behaviour. Each supplier will still want to maximize after-tax profit, which will be achieved by maximizing profit before tax. The profit in any market outcome can be estimated by computing the difference between clearing price and marginal cost at each level of production and integrating over all production levels below the cleared level. Taxation can be used as a mechanism to redistribute wealth, or to extract payment for assets (like transmission lines) that are shared between market participants.

In some settings, like electricity markets, suppliers offer supply curves to the market that in equilibrium are marked up over marginal cost. Here the offer curves are revealed to the market auctioneer, but the true marginal costs are private information. In such circumstances the auctioneer can estimate supplier profits based on the difference between clearing price and offered price,

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and tax this *observed profit* at a fixed proportion. Since the offer curve is a choice of the supplier, it can affect the observed profit by its offer, without having too much effect on its actual profit. In the simplest case where demand is certain, a supplier might increase the price on *infra-marginal* units, i.e. units that have offered below the clearing price, and make observed profit very small without affecting the actual profit.

When demand is uncertain, increasing offer prices on inframarginal units must be done carefully since higher offer prices might decrease the probability of being dispatched. In this setting one can use the theory of market distribution functions to derive a SFE that illustrates the incentive to increase prices on inframarginal offers. This analysis draws heavily on the market distribution theory of both uniform-price auctions [3] and pay-as-bid auctions [2].

Our study of taxation has been motivated by a proposed cost-recovery scheme in which the beneficiaries of an electricity transmission line capacity expansion contribute to its cost in proportion to the additional benefits (profit) accrued by each agent. This is slightly different from the model in which all profits are taxed, in the sense that profits that would have accrued in a counterfactual model (with the original line capacity) are exempt from the tax.

The paper is laid out as follows. In the next section we review supply function equilibrium through the lens of market distribution functions. This is used to derive optimality conditions for suppliers who are taxed a fixed proportion of their observed profit. We show that pure-strategy SFE can be computed as long as the tax is not too high. In the extreme case where the tax is 100%, it is easy to see that the payment mechanism becomes a pay-as-bid scheme. It is well known [2] that pure-strategy SFE occur very rarely in these auctions and mixed strategies prevail. Section 3 deals with the conditions for supply function equilibrium in symmetric duopoly when demand is inelastic and additive demand shocks have a uniform distribution. In section 4 we study an SFE model of a line capacity expansion with inelastic demand. The last section concludes the paper.

2 Supply function equilibrium

As shown in [3] the optimal offer curve $p(q)$ for a generator with cost $C(q)$ facing a market distribution function $\psi(q,p)$ will maximize

$$\Pi = \int (qp - C(q))d\psi(q,p).$$

The market distribution function $\psi(q,p)$ defines the probability that a supplier is not fully dispatched if they offer the quantity q at price p. It can be interpreted as the measure of residual demand curves that pass below and to the left of the point (q,p). Suppose we treat p as function of q. Then

$$\Pi = \int_0^{q_m} (qp(q) - C(q)) \left(\frac{\partial \psi(q,p)}{\partial p} p'(q) + \frac{\partial \psi(q,p)}{\partial q} \right) dq$$

The Euler-Lagrange equation that $p(q)$ must satisfy to minimize a general functional

$$\int_0^{q_m} H(q,p,p')dq$$

is

$$Z(q,p) = \frac{d}{dq} H_{p'} - H_p = 0.$$

In the case where the functional is Π we obtain

$$H_{p'} = (qp(q) - C(q))\frac{\partial \psi(q,p)}{\partial p}$$

$$H_p = q\left(\frac{\partial \psi(q,p)}{\partial p}p'(q) + \frac{\partial \psi(q,p)}{\partial q}\right) + (qp(q) - C(q))(\psi_{pp}p'(q) + \psi_{qp}),$$

and

$$\frac{d}{dq}H_{p'} = (p + qp'(q) - C'(q))\frac{\partial \psi(q,p)}{\partial p} + (qp(q) - C(q))(\psi_{pp}p'(q) + \psi_{qp}).$$

This gives

$$\frac{d}{dq}H_{p'} - H_p = (p - C'(q))\frac{\partial \psi(q,p)}{\partial p} - q\frac{\partial \psi(q,p)}{\partial q}$$

which can be identified with the standard Z function of [3].

Suppose that some fraction $\alpha \in (0,1)$ of the profit earned by a generator is paid as tax. When the market clears at quantity q for a generator at price π then the generator receives

$$\begin{aligned} & q\pi - C(q) - \alpha\int_0^q (\pi - p(t))dt \\ = \; & q\pi - C(q) - \alpha q\pi + \alpha\int_0^q p(t))dt = (1-\alpha)(q\pi - C(q)) + \alpha(\int_0^q p(t)dt - C(q)). \end{aligned}$$

This is a convex combination of uniform and pay-as-bid pricing with multiplier α. Thus the total payoff will be

$$\Pi = (1-\alpha)\int (qp - C(q))d\psi(q,p) + \alpha\int (p - C'(q))(1 - \psi(q,p))dq.$$

We can write down the optimality conditions for the problem faced by a generator maximizing Π. These use the scalar field defined by $Z(q,p) = \frac{d}{dq}H_{p'} - H_p$. Thus

$$\begin{aligned} Z(q,p) &= (1-\alpha)((p - C'(q))\psi_p - q\psi_q) - \alpha(1 - \psi(q,p) - (p - C'(q))\psi_p) \\ &= (p - C'(q))\psi_p - (1-\alpha)q\psi_q - \alpha(1 - \psi(q,p)) \end{aligned}$$

The first-order conditions are given by $Z(q,p) = 0$ and global optimality is guaranteed for a monotonic solution to $Z(q,p) = 0$ if $\frac{\partial}{\partial q}Z(q,p) \leq 0$.

3 Symmetric duopoly for inelastic demand

We use the optimality conditions to look for an equilibrium in symmetric duopoly. Suppose the other player offers a smooth supply function $S(p)$, and demand has cumulative probability distribution function F. Then

$$\begin{aligned} \psi(q,p) &= \Pr[h < q + S(p)] \\ &= F(q + S(p)) \end{aligned}$$

and

$$\begin{aligned} Z(q,p) &= (p - C'(q))\psi_p - (1-\alpha)q\psi_q - \alpha(1 - \psi(q,p)) \\ &= (p - C'(q))S'(p)f(q + S(p)) - (1-\alpha)qf(q + S(p)) - \alpha(1 - F(q + S(p))) \end{aligned}$$

Thus $Z = 0$ is equivalent to

$$(p - C'(q))S'(p) = (1-\alpha)q + \alpha\frac{(1 - F(q + S(p))}{f(q + S(p))}. \tag{1}$$

The global optimality conditions (see [2]) are

$$\begin{array}{lll} (p - C'(q))S'(p) - (1-\alpha)q - \alpha\frac{(1-F(q+S(p))}{f(q+S(p))} & \geq 0, & q < S(p) \\ (p - C'(q))S'(p) - (1-\alpha)q - \alpha\frac{(1-F(q+S(p))}{f(q+S(p))} & = 0, & q = S(p) \\ (p - C'(q))S'(p) - (1-\alpha)q - \alpha\frac{(1-F(q+S(p))}{f(q+S(p))} & \leq 0, & q > S(p) \end{array}$$

These can be guaranteed by $\frac{\partial}{\partial q}Z(q,p) \leq 0$ which amounts to

$$-C''(q)S'(p) - (1-\alpha) - \alpha\left[\frac{(1 - F(q + S(p))}{f(q + S(p))}\right]_q \leq 0.$$

3.1 Examples

Suppose $F(t) = t$ is uniform on $[0,1]$ and $\alpha \leq \frac{1}{2}$. Assume a symmetric duopoly where each generator has capacity $\frac{1}{2}$. Then $C''(q)S'(p) \geq 2\alpha - 1$ guarantees that

$$\begin{aligned} -C''(q)S'(p) - (1-\alpha) - \alpha\left[\frac{(1 - F(q + S(p))}{f(q + S(p))}\right]_q &= -C''(q)S'(p) - (1-\alpha) + \alpha \\ &\leq 0 \end{aligned}$$

The first order condition is then enough to give a supply-function equilibrium. Replacing q by $S(p)$ in (1) yields

$$pS'(p) - (1-\alpha)S(p) - \alpha(1 - 2S(p)) = 0$$

This differential equation can be solved using an integrating factor, whereby

$$pS'(p) + (3\alpha - 1)S(p) = \alpha$$

$$p^{3\alpha-1}S'(p) + (3\alpha - 1)p^{3\alpha-2}S(p) = \alpha p^{3\alpha-2}$$
$$\left(p^{3\alpha-1}S(p)\right)' = \alpha p^{3\alpha-2}$$

$$\begin{aligned} S(p) &= ap^{1-3\alpha} + p^{1-3\alpha}\frac{\alpha}{3\alpha - 1}p^{3\alpha-1} \\ S(p) &= \frac{\alpha}{3\alpha - 1} + ap^{1-3\alpha} \end{aligned}$$

A unique equilibrium can be found by imposing a price cap P at which both generators offer their capacity [7].

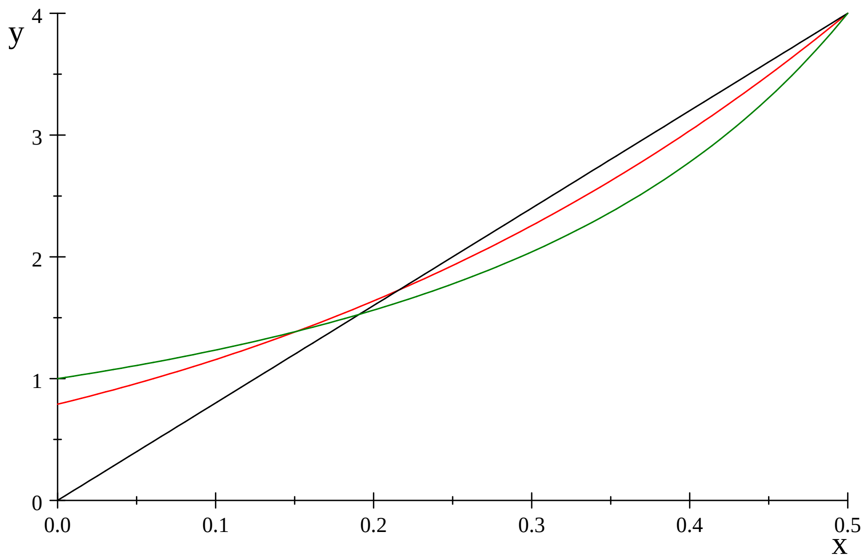


Figure 1: Plot of supply-function equilibrium solutions in a duopoly when inelastic demand is uniformly distributed on [0,1] and each player has capacity 0.5. The red curve is an equilibrium when each supplier must pay 25% of his surplus (above the red curve) as a tax. The green curve gives the equilibrium offer for a 33% tax.

Example 1: Suppose $\alpha = \frac{1}{2}$ and $P = 4$, and each generator has capacity $\frac{1}{2}$. Then

$$S(p) = \frac{\alpha}{3\alpha - 1} + ap^{1-3\alpha}$$

$$\left(\frac{a}{q-1}\right)^2 = p$$

To pass through $(\frac{1}{2}, 4)$ we choose $a = 1$.

Example 2: Suppose $\alpha = \frac{1}{4}$ and $P = 4$, and each generator has capacity $\frac{1}{2}$. Then the solution through $(\frac{1}{2}, 4)$ is

$$q = -1 + \frac{3}{2\sqrt{2}} p^{\frac{1}{4}}$$

$$p = \frac{64}{81}(1+q)^4$$

These solutions are shown in Figure 1.

3.2 Welfare calculations.

We can compute the changes in welfare of each agent from the change in equilibrium. Consider first the case where $\alpha = 0$, and there is a price cap at 4. In perfect competition each generator would offer at price zero and earn no profit. The welfare of the consumers in demand realization h is $4h$, and so the expected competitive total welfare is

$$W = \int_0^1 4h dh = 2$$

In a supply function equilibrium, both players offer linear supply functions $q = \frac{p}{8}$. Thus the total supply is $S(p) = \frac{p}{4}$. When demand is h the clearing price is $4h$, so the total generator profit assuming zero marginal cost is $4h^2$. The expected total generator profit is then

$$\begin{aligned} G &= \int_0^1 4h^2 dh \\ &= \frac{4}{3} \end{aligned}$$

The consumer welfare (assuming a maximum valuation of 4), is

$$\begin{aligned} C &= \int_0^1 h(4-4h)dh \\ &= \frac{2}{3} \end{aligned}$$

Expected total welfare of 2 is divided $\frac{2}{3}$ to the demand and $\frac{2}{3}$ to each generator. The *observed* profit of each generator in demand realization h is defined by the area above their curve, namely the integral of the clearing price at their dispatch $\frac{h}{2}$ minus the offered price at quantity q from $q = 0$ to $\frac{h}{2}$.

$$\begin{aligned} G(h) &= \int_0^{\frac{h}{2}} \left(8\frac{h}{2} - 8q \right) dq \\ &= h^2 \end{aligned}$$

The expected observed profit for both suppliers is then $2\int_0^1 h^2 dh = \frac{2}{3}$, or half their actual profit.

Suppose we now impose a tax by choosing $\alpha = \frac{1}{4}$. Then in equilibrium, each generator offers

$$q = -1 + \frac{3}{2\sqrt{2}} p^{\frac{1}{4}} \tag{2}$$

or

$$p = \frac{64}{81}(1+q)^4. \tag{3}$$

The (before-tax) *observed* profit in demand realization h of each generator if they offer this curve is defined by the area above their curve, namely the integral of the clearing price at their dispatch $\frac{h}{2}$ minus the offered price at quantity q from $q = 0$ to $\frac{h}{2}$. This gives

$$\begin{aligned} G(h) &= \int_0^{\frac{h}{2}} \left(\frac{64}{81}(1+\frac{h}{2})^4 - \frac{64}{81}(1+q)^4 \right) dq \\ &= \frac{4}{405} h^2 \left(2h^3 + 15h^2 + 40h + 40 \right) \end{aligned}$$

The (before-tax) expected observed profit for both suppliers is then

$$2\int_0^1 \frac{4}{405} h^2 \left(2h^3 + 15h^2 + 40h + 40 \right) dh = \frac{128}{243} = 0.527 < \frac{2}{3}$$

which is the total observed profit under a linear supply curve offer. The new offer is arranged to reduce the tax while maintaining a healthy profit. The total tax paid is then $\alpha\frac{128}{243} = \frac{32}{243} < \frac{1}{6}$, the tax collected when linear curves are offered.

Since their costs are zero, the before-tax *actual* profit in demand realization h of each generator if they offer the optimal curve is $\frac{h}{2}\frac{64}{81}(1+\frac{h}{2})^4$. The total expected before-tax *actual* profit for both suppliers is then

$$2\int_0^1 \frac{h}{2}\frac{64}{81}(1+\frac{h}{2})^4 dh = \frac{1586}{1215}$$

The total expected after-tax *actual* profit for both suppliers is

$$T = \frac{1586}{1215} - \frac{32}{243} = \frac{1426}{1215}. \tag{4}$$

Recall that

$$\psi(q,p) = \begin{cases} 0, & q+S(p) \leq 0 \\ q+S(p), & 0 < q+S(p) < 1 \\ 1, & q+S(p) \geq 1 \end{cases}$$

so

$$\psi(q,p(q)) = 2q$$

and

$$\begin{aligned} d\psi(q,p) &= \left(\frac{\partial\psi(q,p)}{\partial q} + \frac{\partial\psi(q,p)}{\partial p}\frac{dp(q)}{dq}\right)dq \\ &= (1+S'(p)p'(q))\,dq \\ &= 2dq \end{aligned}$$

and so the after-tax profit is

$$\begin{aligned} \Pi &= (1-\alpha)\int (qp - C(q))d\psi(q,p) + \alpha\int (p - C'(q))(1-\psi(q,p))dq \\ &= \frac{3}{4}\int (qp - C(q))d\psi(q,p) + \frac{1}{4}\int_0^{\frac{1}{2}} p(q)(1-\psi(q,p))dq \\ &= \frac{3}{4}\int (q\frac{64}{81}(1+q)^4)d\psi(q,p) + \frac{1}{4}\int_0^{\frac{1}{2}} \frac{64}{81}(1+q)^4(1-\psi(q,p))dq \\ &= \frac{3}{4}\int_0^{\frac{1}{2}} (2q\frac{64}{81}(1+q)^4)dq + \frac{1}{4}\int_0^{\frac{1}{2}} \left(\frac{64}{81}(1+q)^4(1-2q)\right)dq \\ &= \frac{713}{1215} \end{aligned}$$

which is half the figure T we computed in (4) as expected.

The welfare of consumers is slightly improved by the tax. Without the tax, generators offer linear supply functions, and the consumer welfare is $\frac{2}{3}$. When a tax is imposed, the generators change their offers, and the price under demand realization h is $\frac{64}{81}(1+\frac{h}{2})^4$. We can then compute the expected total welfare for consumers as

$$\begin{aligned} C &= \int_0^1 h(4 - \frac{64}{81}(1+\frac{h}{2})^4)dh \\ &= \frac{844}{1215} > \frac{2}{3} \end{aligned}$$

The total welfare is then the sum of consumer welfare, generator profit, and tax giving

$$\frac{844}{1215} + \frac{1426}{1215} + \frac{32}{243} = 2.$$

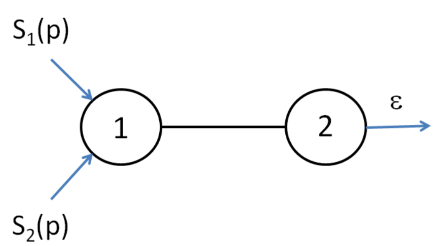
In summary if each generator offers a linear supply curve (as they would in an untaxed equilibrium) then they each earn $\frac{2}{3}$ before tax and $\frac{7}{12}$ after tax, after paying $\frac{1}{12}$ in tax on observed profit of $\frac{1}{3}$. If they instead offer the curve (2) then each generator will appear to earn a profit of $\frac{64}{243}$ but in fact will earn $\frac{793}{1215}$. They will then pay less tax of $\frac{16}{243}$ and each retain a profit of $\frac{793}{1215} - \frac{16}{243} = \frac{713}{1215} > \frac{7}{12}$. The total welfare is 2, and so consumers' welfare increases from $\frac{2}{3}$ to $2 - (2) = \frac{844}{1215}$. The total welfare is consumer welfare plus generator welfare plus tax, giving

$$\frac{844}{1215} + (2)\frac{713}{1215} + (2)\frac{16}{243} = 2$$

So the reaction of the suppliers after the imposition of the tax is to offer to improve their welfare and minimize the tax. The effect of this is to transfer some wealth to consumers.

4 Line capacity expansion

We now consider a model for allocating the costs of a transmission line expansion, by taxing the benefits accruing to generators and demand. The model is a simple two-node network, with symmetric players at one node, and an inelastic demand shock ε at the end of a line that originally has capacity J and expands its capacity to K. Output capacities of firms are denoted $\bar{q}$ and the price cap is $\bar{p}$.



Network example. Line is expanded from K_c to K.

Suppose player 2 offers a supply function $S_2(p)$. If player 1 offers quantity q at price p then the market distribution function is just the probability that either the total quantity offered $q + S_2(p)$ exceeds the link capacity K or that the demand shock ε exceeds the combined offers of the two firms $q + S_2(p)$ at price p;

$$\psi(q,p) = \Pr(q + S_2(p) > \min(K, \varepsilon).$$

If the demand shock is uniformly distributed on $[0, \bar{\varepsilon}]$, then we obtain the following piecewise definition for ψ.

$$\psi(q,p) = \begin{cases} \frac{q+S_2(p)}{\bar{\varepsilon}} & \text{if } q \leq K - S_2(p) \\ 1 & \text{if } q > K - S_2(p). \end{cases} \tag{5}$$

The partial derivatives are $\psi_q = \frac{1}{\bar{\varepsilon}}$ and $\psi_p = \frac{S_2'}{\bar{\varepsilon}} = S_2'\psi_q$when $q \leq K - S_2(p)$ and both are zero otherwise.

4.1 Payoffs

We now look at the payoffs. The actual pre-tax producer surplus if a generator is dispatched q at price π is

$$(\pi - c)q.$$

The system operator observes a different surplus, assuming that $p(q)$ is marginal cost. The observed surplus is then

$$P = \int_0^q (\pi - p(t))dt.$$

There is some ambiguity in the computation of the producer surplus if the offers of two agents are the same and are vertical. We assume that when the offer curves are both vertical the market clears at the smallest price π where $S_1(\pi) + S_2(\pi) = \varepsilon$.

Suppose that producer 1 submits an offer of q at price p. We consider two cases.

Case 1: $S_2(\pi) + q < J$: The perceived surplus as computed from the offer curve $p(q)$ is the same with the line at J as with the line at K. There is no tax imposed.

Case 2: $S_2(\pi) + q > J$: The perceived surplus as computed from the offer curve $p(q)$ is

$$P = \int_0^q (\pi - p(t))dt.$$

The perceived surplus as computed from the offer curve $p(q)$ if the line has capacity J is

$$P' = \int_0^{q'} (p(q') - p(t))dt$$

where q' satisfies

$$S_2(p(q')) + q' = K_c$$

and the tax imposed is $\alpha(P - P')$.

The generator then constructs an offer curve to maximize tax-adjusted profit which is

$$R(q, \pi) = \begin{cases} (\pi - c)\, q, & q < q' \\ (\pi - c)q - \alpha(P - P'), & q > q' \end{cases}$$

Now

$$\begin{aligned} (\pi - c)q - \alpha(P - P') &= (\pi - c)q - \alpha \int_0^q (\pi - p(t))dt + \alpha \int_0^{q'} (p(q') - p(t))dt \\ &= (\pi - c)q - \alpha \left(\pi q - \int_0^q p(t)dt - q'p(q') + \int_0^{q'} p(t)dt \right) \\ &= (1 - \alpha)(\pi - c)q + \alpha \left(q'p(q') + \int_{q'}^q p(t)dt - cq \right) \\ &= (1 - \alpha)\pi q + \alpha \left(q'p(q') + \int_{q'}^q p(t)dt \right) - cq \end{aligned}$$

Thus

$$R(q, \pi) = \begin{cases} \pi q - cq, & q < q' \\ (1 - \alpha)\pi q + \alpha \left(q'p(q') + \int_{q'}^q p(t)dt \right) - cq, & q > q' \end{cases}.$$

Writing p for π, the objective is

$$\begin{aligned}
\Pi &= \int_0^{q'} (p-c)\, q d\psi(q,p) \\
&\quad + \int_{q'}^{Q} \left[(1-\alpha)pq + \alpha\left(q'p(q') + \int_{q'}^{q} p(t)dt\right) - cq\right] d\psi(q,p) \\
&\quad + \left[(1-\alpha)pq + \alpha\left(q'p(q') + \int_{q'}^{q} p(t)dt\right) - cq\right]_Q (1-\psi(Q,p(Q)))
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_{q'}^{Q} f(q,p)d\psi(q,p) + f(Q,p(Q))(1-\psi(Q,p(Q))) \\
= \;& f(Q,p(Q))\psi(Q,p(Q)) - f(q',p(q'))\psi(q',p(q')) \\
& - \int_{q'}^{Q} \frac{d}{dq}\left(f(q,p)\right)\psi(q,p)dq + f(Q,p(Q)) - f(Q,p(Q))\psi(Q,p(Q)) \\
= \;& f(Q,p(Q)) - f(q',p(q'))\psi(q',p(q')) - \int_{q'}^{Q} \frac{d}{dq}\left(f(q,p)\right)\psi(q,p)dq.
\end{aligned}$$

Setting

$$f = \alpha \int_{q'}^{q} p(t)dt$$

gives

$$\alpha \int_{q'}^{Q} p(q)dq - \alpha \int_{q'}^{Q} p(q)\psi(q,p)dq = \alpha \int_{q'}^{Q} p(q)(1-\psi(q,p))dq.$$

Setting

$$f = \alpha q' p(q')$$

gives

$$\alpha q' p(q')\left(1-\psi(q',p(q'))\right).$$

The profit to be optimized by the generator is therefore

$$\begin{aligned}
\Pi &= \int_0^{q'} [pq - cq]\, d\psi(q,p) \\
&\quad + \int_{q'}^{Q} [(1-\alpha)pq - cq]\, d\psi(q,p) \\
&\quad +\alpha \int_{q'}^{Q} p(1-\psi(q,p))dq \\
&\quad +\alpha q' p(q')\left(1-\psi(q',p(q'))\right) \\
&\quad + [(1-\alpha)p(Q)Q - cQ]\left(1-\psi(Q,p(Q))\right)
\end{aligned}$$

The profit to be optimized can be viewed as follows:

$$
\begin{aligned}
\Pi \;=\; & \int_0^Q pqd\psi(q,p) + p(Q)Q(1-\psi(Q,p(Q))) \\
& -\alpha\left[\int_{q'}^Q pqd\psi(q,p) - \int_{q'}^Q p(1-\psi(q,p))dq\right] \\
& -\alpha p(Q)Q(1-\psi(Q,p(Q))) \\
& +\alpha q' p(q')\,(1-\psi(q',p(q'))) \\
& -\int_0^Q cqd\psi(q,p) - cQ(1-\psi(Q,p(Q)))
\end{aligned}
$$

Where

$$
\int_0^Q pqd\psi(q,p) + p(Q)Q(1-\psi(Q,p(Q)))
$$

is total expected generator revenue, and

$$
\begin{aligned}
& \int_{q'}^Q pqd\psi(q,p) - \int_{q'}^Q p(1-\psi(q,p))dq \\
& +Qp(Q)(1-\psi(Q,p(Q))) \\
& -q'p(q')\,(1-\psi(q',p(q')))
\end{aligned}
$$

is total expected perceived benefit of the extra line capacity, and

$$
\int_0^Q cqd\psi(q,p) + cQ(1-\psi(Q,p(Q)))
$$

is expected variable cost. Figure 2 illustrates these terms.

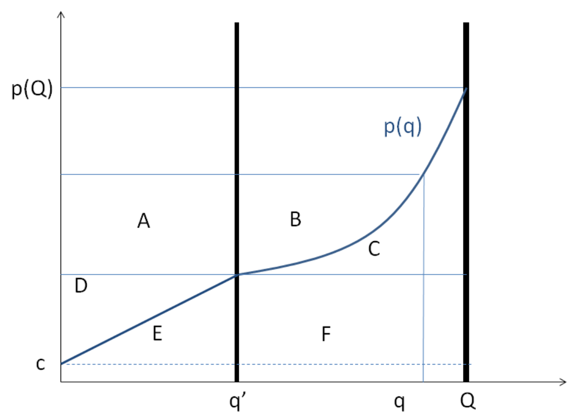


Figure 2: Taxation of net benefits

Total expected generator revenue is area A+B+C+D+E+F integrated over the range from 0 to Q, as well as the revenue that accrues for all realizations where $q = Q$. Total expected perceived benefit of the extra line capacity is the integral of area A+B integrated over the range from $q = q'$ to $q = Q$. We compute this by integrating A+B+C+D+E+F (pq) over the range from q' to Q with density defined by ψ and subtracting the expectation of area F+C (which is $\int_{q'}^Q p(1-\psi(q,p))dq$). We need to lower the total by the expectation of D+E ($q'p(q')$) whenever $q \geq q'$ (which happens with probability $(1-\psi(q',p(q')))$). When $q < q'$ there is no tax counted.

We also need to account for some extra contribution to expected benefits of the line when $q = Q$ (with probability $(1 - \psi(Q, p(Q)))$).

We now consider the case where there is a tax on the benefits of the line. We denote by ψ the market distribution function with the expanded line. The payoff for the offer curve $p(q)$ is then

$$\begin{aligned}
\Pi\left(p\left(q\right)\right) &= \int F\left(q,p,p'\right)dq \\
&= \int_0^{q'} \left(qp\left(q\right) - C\left(q\right)\right)\left(\frac{\partial\psi}{\partial p}p'\left(q\right) + \frac{\partial\psi}{\partial q}\right)dq \\
&\quad + \int_{q'}^{Q} \left((1-\alpha)qp\left(q\right) - C\left(q\right)\right)\left(\frac{\partial\psi}{\partial p}p'\left(q\right) + \frac{\partial\psi}{\partial q}\right)dq \\
&\quad + \alpha\left(\int_{q'}^{Q} p\left(q\right)\left(1 - \psi\left(q,p\left(q\right)\right)\right)dq\right).
\end{aligned}$$

The Z function will have two ranges. For $q < q'$, we have

$$\begin{aligned}
\frac{d}{dq}\left(\frac{\partial F}{\partial p'}\right) &= \left(p + qp' - C'\right)\frac{\partial\psi}{\partial p} + \left(qp - C\right)\left(\frac{\partial^2\psi}{\partial p\partial q} + \frac{\partial^2\psi}{\partial p^2}p'\right) \\
\frac{\partial F}{\partial p} &= q\left(\frac{\partial\psi}{\partial p}p' + \frac{\partial\psi}{\partial q}\right) + \left(qp - C\right)\left(\frac{\partial^2\psi}{\partial p^2}p' + \frac{\partial^2\psi}{\partial p\partial q}\right)
\end{aligned}$$

$$\begin{aligned}
Z\left(q,p\right) &= \frac{d}{dq}\left(\frac{\partial F}{\partial p'}\right) - \frac{\partial F}{\partial p} \\
&= \left(p - C'\left(q\right)\right)\frac{\partial\psi}{\partial p} - q\frac{\partial\psi}{\partial q}
\end{aligned}$$

as we would expect from [3].

For $q > q'$, we have

$$\frac{d}{dq}\left(\frac{\partial F}{\partial p'}\right) = \left((1-\alpha)\left(p + qp'\right) - C'\right)\frac{\partial\psi}{\partial p} + \left((1-\alpha)qp - C\right)\left(\frac{\partial^2\psi}{\partial p\partial q} + \frac{\partial^2\psi}{\partial p^2}p'\right)$$

$$\begin{aligned}
\frac{\partial F}{\partial p} &= (1-\alpha)q\left(\frac{\partial\psi}{\partial p}p' + \frac{\partial\psi}{\partial q}\right) + \left((1-\alpha)qp - C\right)\left(\frac{\partial^2\psi}{\partial p^2}p' + \frac{\partial^2\psi}{\partial p\partial q}\right) \\
&\quad + \alpha\left(1 - \psi\left(q,p\right) - p\frac{\partial\psi}{\partial p}\right).
\end{aligned}$$

Thus

$$
\begin{aligned}
Z(q,p) &= \frac{d}{dq}\left(\frac{\partial F}{\partial p'}\right) - \frac{\partial F}{\partial p} \\
&= \left((1-\alpha)\left(p+qp'\right) - C'\right)\frac{\partial \psi}{\partial p} \\
&\quad -(1-\alpha)q\left(\frac{\partial \psi}{\partial p}p' + \frac{\partial \psi}{\partial q}\right) \\
&\quad -\alpha\left(1-\psi(q,p) - p\frac{\partial \psi}{\partial p}\right) \\
&= \left((1-\alpha)p - C'\right)\frac{\partial \psi}{\partial p} - (1-\alpha)q\frac{\partial \psi}{\partial q} \\
&\quad -\alpha\left(1-\psi(q,p) - p\frac{\partial \psi}{\partial p}\right) \\
&= \left(p - C'\right)\frac{\partial \psi}{\partial p} - (1-\alpha)q\frac{\partial \psi}{\partial q} - \alpha\left(1-\psi(q,p)\right).
\end{aligned}
$$

Observe that this is the same as the Z function for an offer to an auction with a convex combination between uniform and pay-as-bid pricing, as in section 2.

4.2 Example

We now consider an example. The base levels of parameters are as follows:

$g =$	10	maximum shock
$c =$	1	marginal cost
$K =$	8	enlarged line capacity
$J =$	2	restricted line capacity
$\alpha =$	$\frac{1}{4}$	tax rate
$r =$	6	price cap (i.e $\bar{p}$)

The first order condition is

$$
\left(p - C'\right)\frac{\partial \psi}{\partial p} - (1-\alpha)q\frac{\partial \psi}{\partial q} - \alpha\left(1-\psi(q,p)\right) = 0.
$$

Here

$$
\psi(q,p) = \begin{cases} \frac{q+S_2(p)}{g} & \text{if } q \le K - S_2(p) \\ 1 & \text{if } q > K - S_2(p). \end{cases}
$$

so

$$
\frac{\partial \psi}{\partial p} = \frac{S_2'(p)}{g}, \quad \frac{\partial \psi}{\partial q} = \frac{1}{g}
$$

Replacing $S_2(p)$ and q by $S(p)$ in (1) yields

$$
\frac{1}{g}(p-c)S'(p) - \frac{1}{g}(1-\alpha)S(p) - \alpha(1 - \frac{2S(p)}{g}) = 0
$$

or

$$
(p-c)S'(p) - (1-\alpha)S(p) - \alpha(g - 2S(p)) = 0
$$

can be solved using an integrating factor, whereby

$$(p-c)S'(p)+(3\alpha-1)S(p)=\alpha g$$

$$(p-c)^{3\alpha-1}S'(p)+(3\alpha-1)(p-c)^{3\alpha-2}S(p)=\alpha g(p-c)^{3\alpha-2}$$
$$\left((p-c)^{3\alpha-1}S(p)\right)'=\alpha g(p-c)^{3\alpha-2}$$

$$\begin{aligned}
S(p) &= k(p-c)^{1-3\alpha}+(p-c)^{1-3\alpha}\frac{\alpha g}{3\alpha-1}(p-c)^{3\alpha-1}\\
S(p) &= k(p-c)^{1-3\alpha}-\frac{\alpha g}{1-3\alpha}
\end{aligned}$$

To pass through the price cap we require

$$\begin{aligned}
k(r-c)^{1-3\alpha}-\frac{\alpha g}{1-3\alpha} &= \frac{K}{2}\\
k &= \left(\frac{K}{2}+\frac{\alpha g}{1-3\alpha}\right)(r-c)^{3\alpha-1}
\end{aligned}$$

$$S(p)=\begin{cases}\left(\frac{K}{2}+\frac{\alpha g}{1-3\alpha}\right)\left(\frac{p-c}{r-c}\right)^{1-3\alpha}-\frac{\alpha g}{1-3\alpha} & \text{if} \quad p\geq t\\ \frac{J}{2}\frac{p-c}{t-c} & \text{if} \quad p<t\end{cases}$$

where setting $p=t$ makes this function continuous at $q=\frac{J}{2}$.

Observe that the exponent of $\left(\frac{p-c}{r-c}\right)$ vanishes when $\alpha=\frac{1}{3}$. Then the differential equation

$$(p-c)S'(p)+(3\alpha-1)S(p)=\alpha g$$

becomes

$$S'(p)=\frac{\alpha g}{(p-c)}$$

so

$$S(p)=\alpha g\log\frac{p-c}{r-c}+\frac{K}{2}.$$

The section of the supply curve below $q=\frac{J}{2}$ is found by solving $S(t)=\frac{J}{2}$, for t, and setting

$$S(t)=\frac{J}{2}\frac{p-c}{t-c}, \quad p<t.$$

The supply-function equilibria for different choices of α are plotted below.

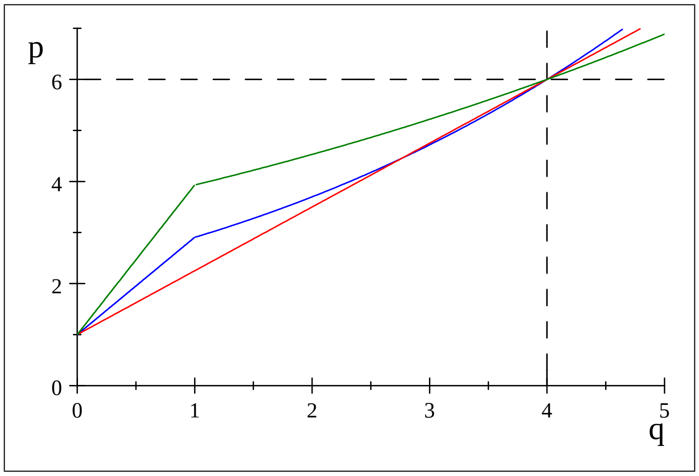
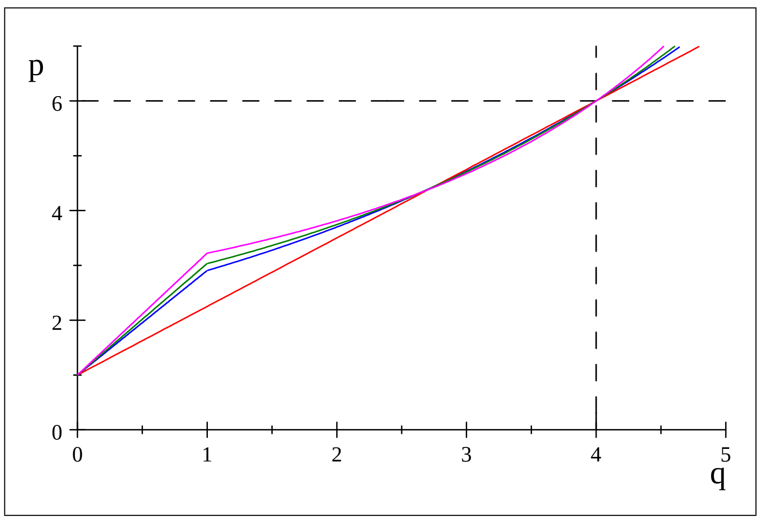


Figure 3: Plot of untaxed equilibrium offer (red) and taxed equilibrium offer (blue) when maximum demand is 10. Equilibrium offer when demand is 20 is shown in green.



Plot of untaxed equilibrium offer (red) and taxed equilibrium offers when maximum demand is 10 and $\alpha = \frac{1}{4}$ (blue), $\alpha = \frac{1}{3}$ (green), and $\alpha = \frac{1}{2}$ (magenta).

The degree to which the taxed equilibrium is marked up above the untaxed equilibrium depends on the range of the demand shock. If the range of the demand shock is large, then there is a high probability that the expanded line will be congested, which even so provides significant bemefits compared with the unexpanded line. This means that the equilibrium offers try to avoid taxation of these by flattening the offer curve. This can be observed in Figure 3.

We finish this example by computing the equilibrium for $\alpha = \frac{1}{4}$, $g = 10$, and different values of J. These are shown in Figure 4. Observe that for small increases in line capacity (from $J = 6$ to $J = 8$) the magenta and red curves almost coincide, so there is minimal change in offer strategy to avoid the tax.

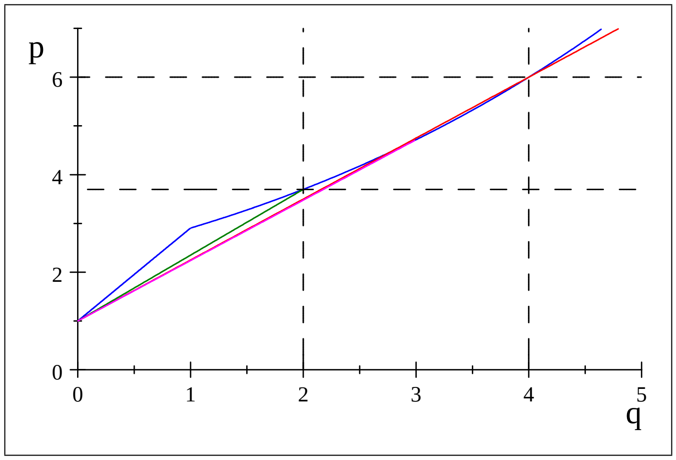


Figure 4: Plot of untaxed equilibrium offer (red) and taxed equilibrium offer when $J = 2$ (blue), $J = 4$ (green), $J = 6$ (magenta).

5 Conclusions

The supply-function equilibrium models outlined in this paper show that taxes imposed on electricity suppliers do not necessarily lead to less competitive outcomes. It is interesting to speculate whether these results remain true for increases in the number of players, asymmetry in suppliers, and contracting. Since they lead to lower overall welfare for suppliers, one might conjecture that some recovery of these losses will be achieved possibly by some out-of-market mechanism. However that is outside the scope of this work.

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5.1 Appendix: Plot of Z for example

In this appendix we verify the equilibrium candidate of section 4 by plotting the Z function. Given

$$S(p) = \begin{cases} \left(\frac{K}{2} + \frac{\alpha g}{1-3\alpha}\right)\left(\frac{p-c}{r-c}\right)^{1-3\alpha} - \frac{\alpha g}{1-3\alpha} & \text{if} \quad p \geq t \\ \frac{J}{2}\frac{p-c}{t-c} & \text{if} \quad p < t \end{cases}$$

we get

$$S'(p) = \begin{cases} \frac{7}{10\left(\frac{1}{5}p - \frac{1}{5}\right)^{\frac{3}{4}}} & \text{if} \quad p \geq t \\ \frac{38\,416}{73\,205} & \text{if} \quad p < t \end{cases}$$

$$\begin{aligned} \frac{\partial \psi}{\partial p} &= \frac{S'(p)}{g} \\ \frac{\partial \psi}{\partial q} &= \frac{1}{g} \end{aligned}$$

We can now plot contours of

$$\begin{aligned} Z(q,p) &= (p-c)\frac{\partial \psi}{\partial p} - (1-\alpha)q\frac{\partial \psi}{\partial q} - \alpha\left(1 - \psi\left(q,p\right)\right) \\ &= (p-c)\frac{7}{10g\left(\frac{1}{5}p - \frac{1}{5}\right)^{\frac{3}{4}}} - (1-\alpha)q\frac{1}{g} - \alpha\left(1 - \frac{q + \left(\frac{K}{2} + \frac{\alpha g}{1-3\alpha}\right)\left(\frac{p-c}{r-c}\right)^{1-3\alpha} - \frac{\alpha g}{1-3\alpha}}{g}\right) \end{aligned}$$

which are presented in Figure 5. These show that $Z(q,p)$ is a decreasing function of q for any fixed p, which demonstrates that the blue curve is an optimal supply function response to itself.

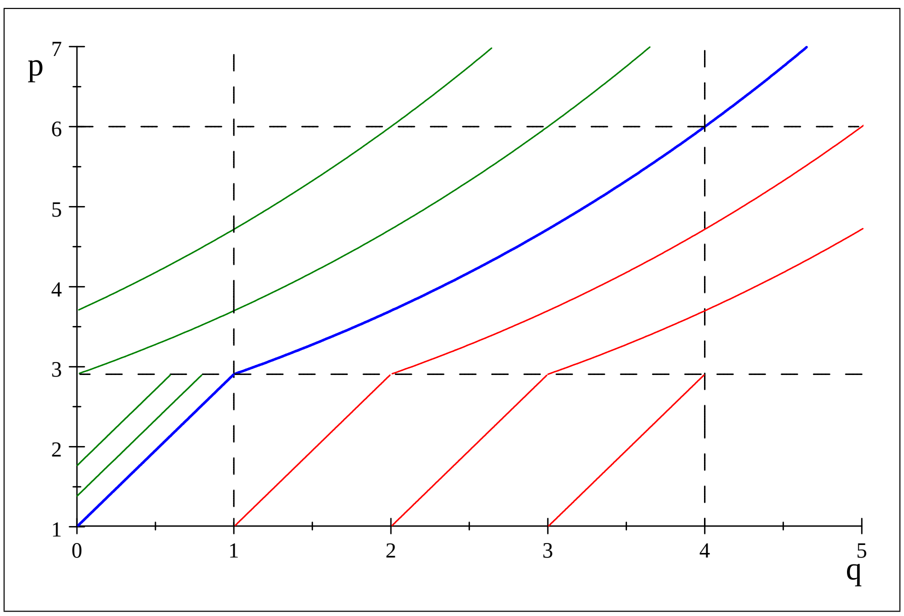

Figure 5: The blue curve is $Z = 0$. Green curves show where $Z > 0$ and red curves where $Z < 0$.